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J. Phys. A: Math. Gen., Vol. 12, No. 9, 1979. Printed in Great Britain

Quantum statistics in a Robertson–Walker universe†

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Received 22 November 1978, in final form 15 January 1979

Abstract. The approach to equilibrium is discussed for a heat bath made of scalar, massless particles in a $k = 0$ Robertson–Walker background which drives a harmonic oscillator.

1. Introduction

It is known (Israel 1972) that matter in a gravitational field can in general only be in an equilibrium configuration if the underlying space–time manifold is stationary. An exception to this rule, recently discussed in this journal by Kennedy (1978), is afforded by massless particles for which only conformal stationarity is required. This enables one to set up the formalism of finite-temperature quantum field theory on Robertson–Walker (RW) universes (see Gibbons and Perry 1978, Kennedy 1978 and references therein).

This paper concerns itself with an application of this machinery to a non-equilibrium situation where a harmonic oscillator interacts with this 'heat bath' made of zero rest mass particles in a RW background. It can be viewed as a generalisation of the well known work by Ford *et al* (1965) on chains of oscillators. The chosen coupling is such that the oscillator 'does not follow the expansion (contraction) of the universe' (like is the case for, say, an antenna trying to detect the cosmic microwave background). Physically one expects that as long as the universe does not expand (contract) rapidly on the scale of a typical collision time of the system, the classical results on approach to equilibrium in the various limits (Ford *et al* 1965) remain valid, but with $T(t) = (R(0)/R(t))T(0)$—the 'Tolman temperature'—playing the role of temperature. At least in the classical limit this expectation is borne out by the computations.

2. The free field

The line element for a $k = 0$ RW universe in the usual coordinates is

$$ds^2 = dt^2 - R^2(t)\, d\boldsymbol{x}^2. \qquad (1)$$

Introducing $\tau = f(t) = \int^t dt'/R(t')$ as the new time coordinate makes it manifest that (1) is conformally Minkowskian, in particular conformally stationary. We consider a massless conformally invariant scalar field on this RW metric:

$$(\Box + \tfrac{1}{6}R)\phi = 0. \qquad (2)$$

† Supported by a stipend from the Bundesministerium für Wissenschaft und Forschung.

Taking advantage of conformal invariance the thermal two-point function for the quantum version of (2) can be written as

$$\langle T\{\phi(\boldsymbol{x}, t), \phi(\boldsymbol{x}, t')\}\rangle = \frac{1}{R(t)R(t')} F_\beta(\boldsymbol{x}, \tau; \boldsymbol{x}', \tau') \tag{3}$$

where F_β is the corresponding two-point function in Minkowski space which satisfies

$$\Box_\eta F_\beta(\boldsymbol{x}, \tau; \boldsymbol{x}', \tau') = -\mathrm{i}\delta(\tau - \tau')\delta^3(\boldsymbol{x} - \boldsymbol{x}') \tag{4}$$

and is given by (see, e.g., Dolan and Jackiw 1974)

$$\begin{aligned} F_\beta(\boldsymbol{x}, \tau; \boldsymbol{x}', \tau') = &-\frac{\mathrm{i}\hbar}{4\pi}\delta((x - x')^2) - \frac{\hbar^2}{4\pi^2}\frac{1}{(x - x')^2} \\ &+ \frac{\hbar}{2\pi^2}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}\int_0^\infty \mathrm{d}p \frac{\sin p|\boldsymbol{x} - \boldsymbol{x}'| \cos p|\tau - \tau'|}{\exp(\hbar p\beta - 1)} \end{aligned} \tag{5}$$

where $(x - x')^2 = (\tau - \tau')^2 - |\boldsymbol{x} - \boldsymbol{x}'|^2$. The first two terms on the RHS of (5) are vacuum contributions. In the context of the model treated presently they should, suitably regularised, give finite corrections to the physical constants appearing therein. Since, however, we are not interested in this effect for the present purposes, we simply discard these terms. To identify β in (3) and (5), we look at the expectation value of the energy density (vacuum terms discarded) at some initial time $t = 0$ and compare with the Stefan–Boltzmann law. This gives $\beta = 1/TR(0)(k_{\mathrm{Boltzmann}} = 1)$ where T is the initial temperature. Using the same argument for arbitrary t shows that the energy density is equal to the one of black-body radiation at temperature $T(t) = R(0)T/R(t)$ in accordance with standard red-shift arguments (Sciama 1971).

3. The model

The model we use was first studied by Aichelburg and Beig (1977). The equations are

$$(\Box + \tfrac{1}{6}R)\phi(\boldsymbol{x}, t) = \lambda \frac{\delta^{(3)}(\boldsymbol{x})}{R^3(t)} Q(t) \tag{6}$$

$$\ddot{Q}(t) + \omega_0^2 Q(t) = \lambda\phi(\boldsymbol{0}, t) \qquad (\dot{Q} \doteq \mathrm{d}Q/\mathrm{d}t). \tag{7}$$

The factor $1/R^3(t)$ on the RHS of (7) makes sure that the 'charge' distribution is the point-limit of a finite one for which the invariant size is constant in time. This means that the oscillator 'does not follow the expansion of the universe'.

The equal-time commutation relations at time t for the quantum versions of (6) and (7) are

$$\begin{aligned} &[\Phi(\boldsymbol{x}, t), \dot{\Phi}(\boldsymbol{x}', t)] = \mathrm{i}\frac{\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')}{R^3(t)} \qquad [Q, \dot{Q}] = \mathrm{i} \\ &[\Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}')] = [\dot{\Phi}(\boldsymbol{x}), \dot{\Phi}(\boldsymbol{x}')] = [\Phi, Q] = [\dot{\Phi}, Q] = [\Phi, \dot{Q}] = [\dot{\Phi}, \dot{Q}] = 0. \end{aligned} \tag{8}$$

One shows that these relations are consistent with the dynamics, i.e. propagated by the equations of motion. Due to the linearity of (6) and (7) the classical and quantum solutions to the Cauchy problem are formally identical. They are given for $t>0$ by

$$Q(t)=(\dot{G}(t)+2\Gamma G(t))Q(0)+G(t)\dot{Q}(0)+\lambda\int_0^t \mathrm{d}t'\,\frac{G(t-t')}{R(t')}\Phi_{\mathrm{H}}(\mathbf{0},t') \tag{9}$$

where

$$G(t)=\exp(-\Gamma t)\sin(\omega t)/\omega, \qquad \Gamma=\lambda^2/8\pi \qquad \omega=+\sqrt{\bar{\omega}^2-\Gamma^2}.$$

$\bar{\omega}^2$ is the 'physical' spring constant which contains an (in our case infinite) renormalisation due to the scalar self-energy of the oscillator. $\Phi_{\mathrm{H}}(\boldsymbol{x},t)$ is the solution of (2) for initial data $\Phi(\boldsymbol{x},0)$, $\dot{\Phi}(\boldsymbol{x},0)$. The expression for $\Phi(\boldsymbol{x},t)$ will not be needed here.

We require the state of the system at $t=0$ to be thermal with respect to the field variable in the sense of (3) and (5) and arbitrary for the oscillator. Oscillator and field should be uncorrelated initially, which means that expectation values containing products of Φ and Q variables at $t=0$ are equal to the corresponding product of expectation values. We are interested in expressions like $\langle Q^2(t)\rangle$. Confining our attention to times $t\gg\Gamma^{-1}$, the exponential terms in (9) can be neglected. Using our assumptions about the nature of the initial state and the fact that $\langle\Phi_{\mathrm{H}}(\boldsymbol{x},0)\rangle_\beta=0$ we are left with

$$\langle Q^2(t)\rangle=\lambda^2\frac{\hbar}{2\pi^2}\int_0^t \mathrm{d}t'\int_0^t dt''\,\frac{G(t-t')}{R(t')}\,\frac{G(t-t'')}{R(t'')}\int_0^\infty \mathrm{d}p\,\frac{p\cos p[f(t')-f(t'')]}{\exp(\hbar p/TR(0))-1}. \tag{10}$$

I $R(t)=\text{constant}$, (10) reduces to the expression found by Ford *et al* (1965). In the general case we obtain a simple answer in the classical limit $\hbar\to 0$, where

$$\langle Q^2(t)\rangle=4\Gamma T\int_0^t \mathrm{d}t'\,\frac{G^2(t-t')}{R(t')}R(0). \tag{11}$$

If $R(t)$ increases monotonically and varies little over a time scale of order $1/\Gamma$, we can replace the RHS of (11) by

$$4\Gamma T\frac{R(0)}{R(t)}\int_0^t \mathrm{d}t'\,G^2(t-t'),$$

which in turn is given for $t\gg 1/\Gamma$ by

$$\frac{T}{\bar{\omega}^2}\frac{R(0)}{R(t)}=\frac{T(t)}{\bar{\omega}^2}.$$

In a similar manner one obtains $\langle\dot{Q}^2(t)\rangle\to T(t)$. Therefore the oscillator energy $\frac{1}{2}\dot{Q}^2+\frac{1}{2}\bar{\omega}^2Q^2$ approaches $T(t)$. Since a classical harmonic oscillator at temperature T has energy equal to T, this is exactly what is required by the zeroth law. Expectation values for higher powers of Q, $\dot{Q}$ can be handled in an analogous fashion.

Acknowledgments

I am indebted to Dr D Dubin who suggested a problem of this sort some time ago. Dr P C Aichelburg kindly read the manuscript. Many thanks are due to Dr D W Sciama for helpful conversions and his kind hospitality at the Department of Astrophysics in Oxford where this work was done. I also thank the referee for suggesting improvements of the original manuscript.

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